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Unavoidable topological minors of infinite graphs

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ABSTRACT

A graph G is *loosely-c*-connected, or ℓ-c-connected, if there exists a number d depending on G such that the deletion of fewer than c vertices from G leaves precisely one infinite component and a graph containing at most d vertices. In this paper, we give the structure of a set of ℓ-c-connected infinite graphs that form an unavoidable set among the topological minors of ℓ-c-connected infinite graphs. Corresponding results for minors and parallel minors are also obtained.

1. Introduction

In this paper, we explore unavoidable topological minors in ℓ-c-connected infinite graphs, building on König's Infinity Lemma, which is stated as follows.

Lemma 1.1. *If G is a connected infinite graph, then G contains a vertex of infinite degree or a one-way infinite path.*

The work in this paper is Ramsey-theoretic in nature. In this paper, we will extend König's Infinity Lemma by identifying unavoidable structures in better connected infinite graphs. We will prove a stronger form of an infinite graph result by Oporowski, Oxley, and Thomas from 1993 found in [4], which we state later as Theorem 1.2(b) and prove independently. In their work, they give the set of unavoidable minors of ℓ-c-connected graphs, which follows a corollary from each of our two main results.

One existing generalization of Lemma 1.1 that we will use is a Menger-type theorem for infinite locally finite graphs proved by Halin in [3]. This theorem is stated as Theorem 3.3 and states that, in a graph with no vertex of infinite degree, the number of independent infinite one-way paths is at least equal to the connectivity of the infinite graph.

Since we only consider vertex connectivity in this paper, we restrict our attention to simple graphs. We say that a graph is *connected* if every pair of vertices is contained in a path in the graph. As stated in the abstract, an infinite graph G is *loosely-c-connected*, or *ℓ-c-connected* if there exists a number d depending on G such that the deletion of fewer than c vertices from G leaves precisely one infinite component and a graph containing at most d vertices. (This notation differs from [4], where ℓ-c-connected graphs are called *essentially c-connected*. We use our abbreviation since e-c-connectivity could be misunderstood as edge connectivity.)

For an edge e in a graph G, we may *contract* e in G, written G/e, by replacing the two ends of e with a single vertex adjacent to every vertex that is adjacent to either end of e in G. A *subdivision* of a graph M is any graph obtained from M by replacing some edges of M with finite paths. We say that a graph M is a *topological minor*, or *series minor*, of a graph G, written $M \preceq_t G$, if G contains a subdivision of M as a subgraph. A graph N is a *minor* of a graph G, written $N \preceq G$, if N can be obtained by contracting a set Y of edges in a subgraph H of G, and N is written H/Y. A graph P is a *parallel minor* of a graph G, written

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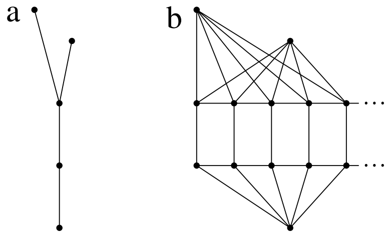

Fig. 1. (a) Tree T. (b) The expansion of T.

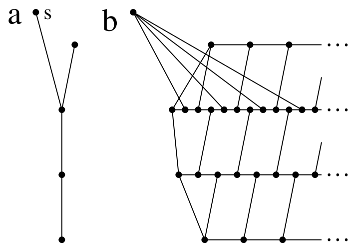

Fig. 2. (a) Tree T. (b) A series expansion of T.

$P \preceq_{\|} G$, if P can be obtained from G by contracting edges. We note that parallel minor is the matroid dual operation of series minor. Parallel minor is related to *induced minor*, which is obtained from a graph by deleting vertices and contracting edges. A parallel minor is an induced minor, and an induced minor is a minor. All graph terminology and notation not defined here follow [1].

A *ray* is a one-way infinite path. A *star* is a vertex u and an infinite vertex set V together with edge set $\{uv : v \in V\}$. A *fan* is the graph of a vertex adjacent to each vertex in a ray. A *ladder* on two rays Y and Z is the graph consisting of the disjoint rays $Y = y_1y_2y_3\dots$ and $Z = z_1z_2z_3\dots$, and edges $y_1z_1, y_2z_2, y_3z_3, \dots$. If the edges $y_2z_1, y_3z_2, \dots$ are added to this ladder, then the result is a *zigzag ladder* on rays Y and Z.

The details of the infinite graphs that we identify as unavoidable minors can be completely expressed as finite trees. We now define the *expansion* of a finite tree T. If T has one vertex then its expansion is a ray. If T has two vertices then its expansion is a fan. These are the two special cases of expansion. A *leaf* is a vertex with degree one. If T has three or more vertices, then let $t_1, t_2, \dots, t_m$ be its leaves and $t_{m+1}, t_{m+2}, \dots, t_n$ be its internal vertices. Then the expansion of T is the graph consisting of vertices $s_1, s_2, \dots, s_m$ and rays $R_{m+1}, R_{m+2}, \dots, R_n$, with a ladder on rays R_i and R_j exactly when $t_it_j \in E(T)$, and a fan on vertex s_k and ray R_l exactly when $t_kt_l \in E(T)$. We say that $s_1, s_2, \dots, s_m$ are the *stars of the expansion* and $R_{m+1}, R_{m+2}, \dots, R_n$ are the *rays of the expansion*. When we refer to the rays of the expansion, we mean these particular rays.

An example of expansion is given in Fig. 1, where tree T in Fig. 1(a) is expanded in Fig. 1(b).

The graph $K_{c,\infty}$ is the infinite bipartite graph containing an independent set A with c vertices and an infinite independent set B, such that $A \cup B = V(K_{c,\infty})$ and each vertex in A is adjacent to every vertex in B. Note that $K_{1,\infty}$ is a star. We add an edge between each pair of the c vertices in A to $K_{c,\infty}$ to obtain the graph $K'_{c,\infty}$.

The countable version of part (b) of the following theorem is proved in [4]; part (a) is mentioned without proof.

Theorem 1.2. *For each positive integer c, let $\mathcal{M}_c$ be the set of graphs that consists of $K'_{c,\infty}$ and expansions of c-vertex trees. Then the following hold.*

(a) *Every graph in $\mathcal{M}_c$ is ℓ-c-connected.*
(b) *Every ℓ-c-connected graph has a minor that is isomorphic to a graph in $\mathcal{M}_c$.*
(c) *No graph in $\mathcal{M}_c$ contains another graph in $\mathcal{M}_c$ as a minor.*

Note that Theorem 1.2 completely characterizes all unavoidable (or minimal) minors of ℓ-c-connected graphs, generalizing König's Infinity Lemma. In this paper, we actually prove two stronger results, each of which has Theorem 1.2(b) as a corollary.

To state our main result we first define a *series expansion* of (T, S), where T is a finite tree, S is a set of leaves of T, and $S \neq V(T)$. Note that S may be empty. A series expansion is basically a subgraph of an expansion of T, except that leaves not in S correspond to rays. The reader may choose to skip the following detailed definition since the idea is clearly illustrated in Fig. 2.

For the purpose of avoiding notation clutter, we first describe a graph G, from which we will obtain the series expansion of (T, S). Let $V(T) = \{t_1, t_2, \dots, t_n\}$ with $S = \{t_1, t_2, \dots, t_m\}$. Let $R_i = r^i_1 r^i_2 \dots$ be a ray for $i = m+1, m+2, \dots, n$. Then G is constructed from vertices $s_1, s_2, \dots, s_m$, and disjoint rays $R_{m+1}, R_{m+2}, \dots, R_n$ by adding edges $s_ir^j_i, s_ir^j_{i+n}, s_ir^j_{i+2n}, \dots$, for each $t_it_j \in E(T)$ such that $i \le m < j$, and edges $r^i_j r^j_i, r^i_{j+n} r^j_{i+n}, r^i_{j+2n} r^j_{i+2n}, \dots$, for each $t_it_j \in E(T)$ such that $i, j > m$. Notice that G

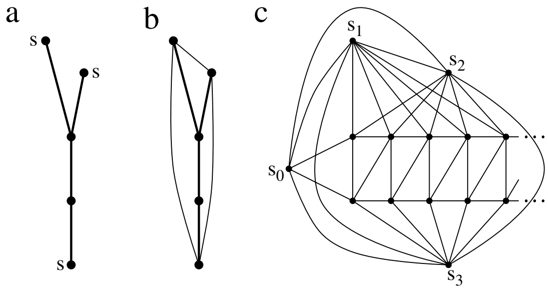

Fig. 3. (a) Tree T with leaves S. (b) Graph $H \supseteq T$. (c) An expansion of (H, S).

may have many vertices of degree at most two, all of which are incident only with edges in the rays. The graph obtained from G by contracting, one by one, the edges incident with a vertex of degree at most two is the *cosimplification* of G, which we call a *series expansion* of (T, S). Note that the resulting series expansion depends not only on T and S, but also on how vertices of T are labelled. It is straightforward to verify that all series expansions of the pair (T, S) are *series-equivalent*, meaning that any one contains the other as a series minor. We will refer to vertices in S and $V(T) - S$ as *star vertices* and *ray vertices*, respectively.

In addition to series expansions of trees, we also need to define different versions of $K_{c,\infty}$. A tree is *branching* if it has no vertices of degree two. Let T be a finite branching tree with exactly c leaves, labelled $1, 2, \ldots, c$, where c is at least three. The *duplication* of T is obtained by taking infinitely many disjoint copies of T and identifying the leaves that have the same label. Note that the duplication of $K_{1,c}$ is exactly $K_{c,\infty}$. For $c = 1, 2$, we will also consider $K_{1,c}$ a branching tree with c leaves, and we define its duplication to be $K_{c,\infty}$. Each duplication of a branching tree with c leaves is a *version of* $K_{c,\infty}$.

For each positive integer c, let $\mathcal{T}_c$ be the set of graphs that consists of duplications of branching trees with c leaves and series expansions of (T, S) with $|T| = c$. The following is the main result in this paper, which characterizes a complete set of unavoidable topological minors of ℓ-c-connected graphs.

Theorem 1.3. *The following hold for every positive integer c.*

(a) *Every graph in $\mathcal{T}_c$ is ℓ-c-connected.*
(b) *Every ℓ-c-connected graph has a topological minor that is isomorphic to a graph in $\mathcal{T}_c$.*
(c) *If $M, N \in \mathcal{T}_c$ and $N \preceq_t M$, then M and N are series-equivalent and are isomorphic to the same duplication of $K_{c,\infty}$ or are series expansions of a pair (T, S).*

Note that 1.3(c) states that nonequivalent graphs in $\mathcal{T}_c$ are not comparable, which means that, up to equivalence, there is no redundancy in $\mathcal{T}_c$. We could define $\mathcal{T}_c$ by taking one representative from each equivalence class, which would give rise to a formulation similar to 1.2(c). Since no natural representatives are available, we leave the formulation as it is.

Our final result is a similar theorem on parallel minors. Since no vertex or edge deletions are allowed, the unavoidable structures will be expansions of graphs, instead of trees, and the expansions are consequently more complex. We will require a finite graph to have a leaf-maximal spanning tree. A spanning tree T of a finite graph is called *leaf-maximal* if the graph has no spanning tree such that its set of leaves properly contains the set of leaves of T.

We consider pairs (H, S), where H is a connected finite graph and S is a vertex set contained in $V(H)$. Recall that $H[S]$ is the subgraph H induces on S. If H has one or two vertices, we require that $|S| = |H| - 1$, and we define an *expansion* of (H, S) to be a ray or a fan, respectively. If H has three or more vertices, we require that $H - S$ is a tree, $H[S]$ is a clique, and H has a leaf-maximal spanning tree with S as its set of leaves. Let $S = \{t_1, t_2, \ldots, t_m\}$ and $V(H) - S = \{t_{m+1}, t_{m+2}, \ldots, t_n\}$. An *expansion of* (H, S) is a graph consisting of vertices $s_0, s_1, s_2, \ldots, s_m$ and rays $R_{m+1}, R_{m+2}, \ldots, R_n$, with a zigzag ladder on rays R_i and R_j exactly when $t_it_j \in E(H)$, a fan on vertex s_k and ray R_l exactly when $t_kt_l \in E(H)$, an edge between each pair of vertices in $\{s_0, s_1, \ldots, s_m\}$, and an edge between s_0 and the first vertex of each ray R_{m+i}. For an example, see Fig. 3. Note that there are two ways to put a zigzag ladder onto a pair of rays, therefore there may be several different graphs that are expansions of a pair. For any pair of graphs G and G' in such a set, $G \cong G'/Y$, where Y consists of initial segments of the rays, so we say that the two graphs G and G' are *parallel-equivalent*.

For each positive integer c, let $\mathcal{P}_c$ be the set of graphs that consists of K_∞, $K'_{c,\infty}$, and expansions of (H, S), over all pairs as described in the last paragraph, such that $|H| = c$. The following is our final result, a characterization of unavoidable parallel minors of ℓ-c-connected graphs.

Theorem 1.4. *The following hold for every positive integer c.*

(a) *Every graph in $\mathcal{P}_c$ is ℓ-c-connected.*
(b) *Every ℓ-c-connected graph has a parallel minor that is isomorphic to a graph in $\mathcal{P}_c$.*
(c) *If $M, N \in \mathcal{P}_c$ and $N \preceq_\parallel M$, then M and N are parallel-equivalent and are both isomorphic to $K'_{c,\infty}$, both isomorphic to K_∞, or expansions of a pair (H, S).*

We point out that this result gives a characterization of the set of unavoidable induced minors of ℓ-c-connected graphs: K_∞ and $K'_{c,\infty}$ together with other members of $\mathcal{P}_c - \{K_\infty, K'_{c,\infty}\}$ with s_0 being deleted.

The rest of the paper is organized as follows. In Section 2, we prove parts (a) and (c) of our main results. In Section 3, we prove a result on augmenting path, which will be used in later analysis. In Sections 4 and 5 we prove Theorems 1.3(b) and 1.4(b), respectively.

2. The qualification of unavoidable sets

We will first show that the unavoidable graphs are ℓ-c-connected. The following simple lemma is a first step in proving this. The proof is straightforward and is omitted.

Lemma 2.1. *Let T be a tree containing c vertices. Then every series expansion of $(T, \emptyset)$ is ℓ-c-connected.*

We now address nonredundancy.

Lemma 2.2. *Every graph in $\mathcal{M}_c \cup \mathcal{T}_c \cup \mathcal{P}_c$ is ℓ-c-connected.*

Proof. Clearly K_∞ and every version of $K_{c,\infty}$ is ℓ-c-connected. Furthermore, a ray is ℓ-1-connected and a fan and a ladder are each ℓ-2-connected. Since graphs in $\mathcal{M}_c \cup \mathcal{P}_c$ are obtained from graphs in $\mathcal{T}_c$ by adding edges, it suffices to show that, for $c \geq 3$, every graph in $\mathcal{T}_c$ is ℓ-c-connected. Take a tree T with c vertices. We apply Lemma 2.1 to conclude each series expansion of $(T, \emptyset)$ is ℓ-c-connected.

We will suppose that each series expansion G of (T, S) is ℓ-c-connected if $|S| = k$, where k is fewer than the number of leaves in T. Take a leaf of T that is a ray vertex and let R be the corresponding ray in G. The vertices $V(R)$ are adjacent with the vertex set of only one other ray of G. We will show that G/R is ℓ-c-connected and conclude by induction on k that every member of $\mathcal{T}_c$ is ℓ-c-connected.

Contract R to a vertex r and let $G' = G/R$. Then take $V' \subset V(G')$, a cut set of G' with fewer than c vertices.

If $r \notin V'$, then V' is also a cut set of G. By induction, G is ℓ-c-connected, which implies that $G \setminus V'$ consists of an infinite component X and a graph H with at most d vertices, where d is a number that depends only on G. As R is in X, $G' \setminus V'$ consists of the infinite component X/R and a graph H. Suppose then that r is in V'. By induction again, $G' - r$ is ℓ-$(c-1)$-connected, so any vertex cut set in $G' - r$ with fewer than $c - 1$ vertices separates $G' - r$ into a component and a graph with at most d' vertices for some integer d' depending on $G' - r$.

The graph $G' \setminus V'$ therefore consists of a component and a graph with at most $\max\{d, d'\}$ vertices, and we conclude that G' is ℓ-c-connected. □

The proof of the following lemma is straightforward and is omitted.

Lemma 2.3. *If P and Q are disjoint rays in a graph G joined by an infinite set Π of independent paths, then G contains a subdivision of a ladder with poles contained in $P \cup Q$, with an infinite subset of Π forming the rungs.*

We will define some terminology for use in proving Theorems 1.3(c), 1.4(c) and 1.2(c). A graph G is *k-disconnected*, for a positive integer k, if there is a set of finite graphs $G_1, G_2, \ldots$ such that G is obtained by identifying V_i, a set of $a_i \leq k$ vertices of G_i, with a_i vertices of G_{i+1} for all positive integers i. Note that, if G is k-disconnected, then it is also k'-disconnected for all $k' > k$. We assume that the edges in $G_i[V_i]$ are identical to the edges in $G_{i+1}[V_i]$. Then G is the *k-path-sum* of $\{G_i\}_{i=1,2,\ldots}$. Since V_i is a cut set for $i = 1, 2, \ldots$, graph G is not ℓ-$(k+1)$-connected.

It is worth noting that each minor G' of G is the k-path-sum of some sequence $\{G'_i\}_{i=1,2,\ldots}$ such that G'_i is obtained from G_i by taking a minor of G_i and possibly identifying some of the vertices in the result for $i = 1, 2, \ldots$. We make the following observation.

Lemma 2.4. *Every minor of a k-disconnected graph is k-disconnected.*

For any ray R, it is not difficult to see that if R meets some V_i then R meets all V_j with $j > i$. Thus, if a graph is k-disconnected, then it does not have $(k+1)$ independent rays.

Let S be the set of vertices in G that are in infinitely many graphs G_i in the k-path-sum. Let $m = k - |S|$. We will use m, k, and S defined here when stating the remaining lemmas in this section and we observe the following.

Lemma 2.5. *For $S' \subseteq S$, the graph $G \setminus S'$ is $k - |S'|$-disconnected.*

Two rays R and R' are *indistinguishable* if $R \setminus P = R' \setminus P'$ for some finite paths P and P'. Two sets of rays $\{R_1, \ldots, R_m\}$ and $\{R'_1, \ldots, R'_m\}$ are indistinguishable if there is a permutation σ such that R_i is indistinguishable from $R'_{\sigma(i)}$ for all i. The following observation is another consequence of our structure.

Lemma 2.6. *Suppose $|V_i| = k$, for all positive integers i, and each graph G_{i+1} contains a unique set of independent paths from the vertices in V_i to the vertices in V_{i+1}. Let $R_1, R_2, \ldots, R_m$ be a set of m independent rays in G. If $R'_1, R'_2, \ldots, R'_m$ are independent rays of M, then $\{R'_1, R'_2, \ldots, R'_m\}$ and $\{R_1, R_2, \ldots, R_m\}$ are indistinguishable.*

We take the assumptions of Lemma 2.6 to hold for the next three lemmas. We will refer to the assumption that each graph G_i contains a unique set of independent paths from the vertices in V_{i-1} to the vertices in V_i as *uniqueness*.

Let X be a set of edges of G. We now consider the graph $G \setminus X$. Take ray R from a set of m independent rays in G. Let $X' = X \cap E(R)$. Suppose $X' = \{e_1, e_2, \cdots\}$ is infinite. Let G_{i_j} be the graph from which e_j is taken, for $j = 1, 2, \ldots$. It is convenient to assume that $i_1 \le i_2 \le \cdots$. By uniqueness, each graph $G_{i_j} - e_j$ contains fewer than m disjoint paths from $V_{i_{j-1}}$ to V_{i_j}. Thus the graph $G_{i_j} - e_j$ contains a cut set with at most $k-1$ vertices. Let V'_2 be the $(k-1)$-vertex cut set in the graph with least index, let V'_3 be the cut set in the graph with next lowest index, and so on. Evidently $G \setminus X'$ may be obtained from some infinite sequence of graphs $G'_1, G'_2, \ldots$ by identifying the vertices V'_j in G'_{j-1} with V'_j in G'_j, for $j = 2, 3, \ldots$. We conclude that $G \setminus X'$ is $(k-1)$-disconnected. By Lemma 2.4, $G \setminus X$ is $(k-1)$-disconnected, and we note the following.

Lemma 2.7. *The deletion of infinitely many edges from any of the m rays in G results in a $(k-1)$-disconnected graph.*

Take m independent rays in G: $R_1, R_2, \ldots, R_m$. Let Q be the set of edges in $G[V(R_1) \cup V(R_2) \cup \cdots \cup V(R_m) \cup S]$ that are not in $E(R_1) \cup E(R_2) \cup \cdots \cup E(R_m)$. Take a set Y of edges in G.

Suppose Y contains an infinite set Y' of edges between two rays R_1 and R_2. Since R_1 and R_2 are contained in $G \setminus S$, no vertex is incident with infinitely many edges in Y', hence Y' contains an infinite set of pairwise non-adjacent edges. By Lemma 2.3, $(R_1 \cup R_2) \cup Y'$ contains a ladder with rung set Y'' contained in Y'. Let the rungs be $e_1, e_2, \ldots$ in the graphs $G_{i_1}, G_{i_2}, \ldots$, respectively, where $i_1 \le i_2 \le \cdots$. By uniqueness, each graph in $G_{i_1}/e_1, G_{i_2}/e_2, \ldots$ contains fewer than m disjoint paths from $V_{i_{j-1}}$ to V_{i_j}. Then each graph G_{i_j}/e_j contains a cut set of G/Y'' with at most $k-1$ vertices, hence G/Y'' is the $(k-1)$-path-sum of a sequence of graphs. Evidently, G/Y'' is $(k-1)$-disconnected, hence, by Lemma 2.4, G/Y is $(k-1)$-disconnected.

Suppose then that Y contains an infinite set Y' of edges between a ray R_1 and a vertex in S, say s. Let $e_1, e_2, \ldots$ be the edges of Y' in the graphs $G_{i_1}, G_{i_2}, \ldots$, respectively, where $i_1 \le i_2 \le \cdots$. By uniqueness, each graph in $G_{i_1}/e_1, G_{i_2}/e_2, \ldots$ contains fewer than m disjoint paths from $V_{i_{j-1}}$ to V_{i_j}. Then each graph G_{i_j}/e_j contains a cut set of G/Y' with at most $k-1$ vertices, and G/Y' is the $(k-1)$-path-sum of a sequence of graphs. Evidently, G/Y' is not ℓ-k-connected. By Lemma 2.4, G/Y is $(k-1)$-disconnected. We make the following observation.

Lemma 2.8. *If set $Y \cap Q$ is infinite then G/Y is $(k-1)$-disconnected.*

Let G_Y be the subgraph of G induced by the vertices incident with edges in Y. If G_Y contains a path P between two vertices in S, say s_1 and s_2, then let G/P be obtained from G by contracting P to the vertex s'. Since s_1 and s_2 are incident with infinitely many edges, there is some index z such that $G_z, G_{z+1}, \ldots$ all contain s_1 and s_2. Let G'_z be the k-path-sum of $G_1, G_2, \ldots, G_z$. For integer i at least $z+1$, let G'_i be obtained as follows. If P is in G_i, then let G'_i be obtained from G_i by contracting P to vertex s'. Otherwise, let G'_i be obtained from G_i by identifying s_1 and s_2, and relabelling the vertex s'. Clearly s_1 and s_2 are in each of the sets $V_z, V_{z+1}, \ldots$, hence G/P is the $(k-1)$-path-sum of $G'_z, G'_{z+1}, \ldots$, and G/P is $(k-1)$-disconnected. By Lemma 2.4, G/Y is $(k-1)$-disconnected, and we make the following observation.

Lemma 2.9. *If any component of G_Y contains two or more vertices of S, then G/Y is $(k-1)$-disconnected.*

The following proof shows nonredundancy among the members of $\mathcal{T}_c$.

Proof of Theorem 1.3(c). Take integer c and graphs M and N of $\mathcal{T}_c$ such that $N \preceq_t M$. By Theorem 1.3(a), both of these graphs are ℓ-c-connected. Take X and Y in $E(M)$ such that $N = M \setminus X/Y$. Note that each edge in Y is a series element in $M \setminus X$. If M is a version of $K_{c,\infty}$, then it is the duplication of a branching tree T, hence N contains no rays and is also a version of $K_{c,\infty}$. Since T has no proper topological minor containing c leaves, it is an easy exercise to show that N is also the duplication of T, and the theorem holds.

We assume then that M is the series expansion of (T_M, S_M). If $|T_M| = 1$, then M is a ray. The only ℓ-1-connected minor of M then contains a ray; hence N is a ray and the theorem holds. We assume that the theorem holds if $c < k$ for some integer k at least two. Suppose $c = k$. By construction, M is c-disconnected. Furthermore, M satisfies the conditions of Lemmas 2.6 and 2.8.

It is useful to note that, since no star vertices are created by deleting edges and contracting series edges, N does not have more star vertices than M. Since c is the sum of the number of stars in N and the number of rays of the expansion of N, the graph $M \setminus X$ has as many independent rays as M does. Let $m = c - |S_M|$. By Lemma 2.6, the set of m independent rays $R'_1, R'_2, \ldots, R'_m$ in $M \setminus X$ is indistinguishable from the set of rays $R_1, R_2, \ldots, R_m$ of M. Evidently each of the rays $R''_1, R''_2, \ldots, R''_m$ of N can be obtained by contracting edges in a ray of $M \setminus X$. That is, $R''_i = R'_i/Y_i$, for some edge set Y_i in R'_i, for $i = 1, 2, \ldots, m$. Thus $V(R''_i) \subseteq R'_i$ for each i. If $m = c$, then N is the series expansion of a pair (T_N, S_N) and $S_N = \emptyset$. By Lemma 2.8, Y contains finitely many edges that are in no ray of M. Evidently, N contains infinitely many edges between the rays R'_i and R'_j exactly when M contains infinitely many edges between R_i and R_j. We conclude that $T_N \cong T_M$.

We may assume then that $S_M \neq \emptyset$. Take a vertex v in T_M that corresponds to a star vertex s in M. Let w be the vertex in T_M adjacent with v and let R_i be the ray of M corresponding to w. Now M contains a subdivision N' of N. By Lemma 2.5, the graph $M - s$ is $(c-1)$-disconnected, and, by Lemma 2.4, every minor of a $(c-1)$-disconnected graph is $(c-1)$-disconnected. Therefore N' is not a minor of $M - s$, and we conclude that vertex s is in N'. Now $M - s$ contains $N' - s$. Clearly, the cosimplification of $N' - s$ is a topological minor of $M - s$. It follows that the cosimplification of $N' - s$ is a topological minor of the cosimplification of $M - s$, and both of these graphs are members of $\mathcal{T}_{c-1}$. By our induction hypothesis, these two cosimplifications are expansions of the same pair $(T_M - v, S_M - v)$. For edge t_it_j of T_M, let $Q_{t_it_j}$ be the set of edges of M that are between the ray R_i or star vertex s_i and the ray R_j or star vertex s_j. Now $V(R''_i) \subseteq V(R_i)$ for the ray R''_i of the expansion

N, thus N is isomorphic to the graph obtained from M by adding a vertex s and a set of edges from Q_{vw} between s and ray R_i, or N is series-equivalent to it. Since adding only a finite set of edges from Q_{vw} results in a $(c-1)$-disconnected graph, N contains an infinite set of edges in Q_{vw}, hence N is the expansion of (T_M, S_M), as desired. □

We now prove nonredundancy among the members of $\mathcal{P}_c$. An *end* of an infinite graph is an equivalence class of rays, where two rays are said to be in the same end of a graph, or equivalent, exactly when they are joined by infinitely many independent paths.

Proof of Theorem 1.4(c). Let c be a positive integer. Take M and N in $\mathcal{P}_c$ such that $N \preceq_{\|} M$. Take edge set Y in M such that $N = M/Y$. By Theorem 1.4(a), N is ℓ-c-connected. If M is isomorphic to $K'_{c,\infty}$, then N contains no ray, hence $N \cong K'_{c,\infty}$ and the theorem holds.

We assume therefore that M is the expansion of (H_M, S_M). By construction, M is c-disconnected. Furthermore, it satisfies the assumptions of Lemmas 2.6, 2.8 and 2.9. Take a tree T_M that spans H_M and has the vertices of S_M as leaves. Let $m = |H_M| - |S_M|$ and let $\{R_1, R_2, \ldots, R_m\}$ be the rays of the expansion M.

Suppose Y contains the edge set of a ray R'_i contained in a ray R_i corresponding to a vertex t_i in H_M. We first assume that t_i is adjacent to fewer than two vertices in $V(T_M) \setminus S_M$. Since t_i is not a leaf corresponding to a star, it is adjacent to a vertex t_j that corresponds to a star of M. If this star is adjacent with the vertices of a ray R_k of the expansion M, where $k \neq i$, then we replace the edge $t_i t_j$ in T_M with $t_k t_j$ to obtain a spanning tree of H_M whose leaves properly contain the set S_M, which contradicts the leaf-maximality of T_M. If no vertex corresponding to a ray of M other than t_i is adjacent with t_j, then, to contract R'_i in M, we must delete all but finitely many edges between a star of the expansion M and a ray of the expansion. Clearly this deletion results in a $(c-1)$-disconnected graph, and, by Lemma 2.4, N is $(c-1)$-disconnected, a contradiction. Next, we assume that t_i is adjacent with at least two vertices in $V(T_M) \setminus S_M$. We contract R'_i in M to a vertex s_{R_i}. Now M/R'_i has $m-1$ rays. The rays are not all in the same end of M/R'_i, however. Take a cut set V' of M/R'_i consisting of the star vertices and a vertex in each of the rays contained in one end. Clearly V' has fewer than c vertices, and each component of $M/R'_i \setminus V'$ is $(c-1)$-disconnected. Since N is ℓ-c-connected, by Lemma 2.4, it is not a minor of any component of $M/R'_i \setminus V'$. It is also easy to see that it is not a minor of M/R'_i. We conclude with the following observation.

2.9.1. *For each ray R_i, the set $E(R_i) \setminus Y$ is infinite.*

By Lemma 2.8, $Q \cap Y$ is finite, thus $M/(Q \cap Y)$ contains a set of m rays indistinguishable from the rays of M. By 2.9.1, Y contains no ray that is contained in a ray of M, hence M/Y contains a set of m rays that are indistinguishable from the rays of M. Take the rays $\{R'_1, R'_2, \ldots, R'_m\}$ in N and the rays $\{R_1, R_2, \ldots, R_m\}$ of the expansion M such that R'_i is indistinguishable from R_i for each i. Evidently N is not isomorphic to $K'_{c,\infty}$. Furthermore, for each star s_k of M, we take vertex s'_k in N that is s_k or is obtained by contracting the component of G_Y that contains the star s_k. By Lemma 2.9, no component of G_Y contains two star vertices, thus exactly $|S_M|$ vertices of N are identified in this way. Since $Q \cap Y$ is finite, R'_i and R'_j have infinitely many edges between them in N exactly when R_i and R_j do in M. Also, R'_i and s'_k have infinitely many edges between them in N exactly when R_i and s_k do in M. By Lemma 2.6, the rays of the expansion N are indistinguishable from the set $\{R'_1, R'_2, \ldots, R'_m\}$. Thus, if R'_i and R'_j have infinitely many edges between them, then N contains a zigzag ladder on R'_i and R'_j. Furthermore, if R'_i and s'_k have infinitely many edges between them, then s'_k is adjacent with all of the vertices of a ray contained in R'_i. We conclude that N must be the expansion of (H_M, S_M), and the theorem holds. □

We complete this section with a proof of the nonredundancy among the members of $\mathcal{M}_c$.

Proof of Theorem 1.2(c). Take positive integer c, and take $M, N \in \mathcal{M}_c$ such that $N \preceq M$. Observe that $K'_{c,\infty}$ contains no rays, so if M is isomorphic to $K'_{c,\infty}$, then so is N.

Take M in $\mathcal{M}_c - \{K'_{c,\infty}\}$ and tree T such that M is an expansion of T. Let S be the stars of the expansion M and let $R_1, R_2, \ldots, R_m$ be the rays of the expansion M. By construction of the expansion, we may select $G_1, G_2, \ldots$ such that G is the c-path-sum of this sequence of graphs, each graph in the sequence is a tree, and these graphs are all isomorphic. Observe that each graph G_i contains a unique set of independent paths from the c vertices in V_{i-1} to the c vertices in V_i.

Take N in $\mathcal{M}_c$ that is a minor of M. By Theorem 1.2(a), N is ℓ-c-connected. Let $N = M \setminus X/Y$. We apply Lemmas 2.7 and 2.4 to conclude the following.

2.9.2. *$X \cap E(R)$ is finite.*

For an edge $e = t_i t_j$ of T, let Q_e be the set of edges of M that are between the ray R_i and the ray R_j or star vertex s_j. Let $X' = Q_e \cap X$ for some edge e in T. Suppose $Q_e \setminus X'$ is finite. Then, for some integer l, each graph in $G_l, G_{l+1}, \ldots$ in the c-path-sum of M contains an edge in X' that is a cut edge in its respective graph. For each integer n at least l, the edge e_n is a cut edge of the tree G_n and V_n has vertices in each component of $G_n - e_n$. If $M \setminus X'$ has one end, then it is clearly $(c-1)$-disconnected and, by Lemma 2.4, N is not ℓ-c-connected, a contradiction. Then $M \setminus X'$ has multiple ends and we take a cut set V' of $M \setminus X'$ consisting of the star vertices and a vertex in each of the rays contained in one end. Clearly V' has fewer than c vertices, and each component of $(M \setminus X') \setminus V'$ is $(c-1)$-disconnected. Since N is ℓ-c-connected, by Lemma 2.4, it is not a minor of any component of $(M \setminus X') \setminus V'$. It is easy to see that it is also not a minor of $M \setminus X'$. We conclude with the following observation.

2.9.3. *The set $Q_e \setminus X$ is infinite for every edge $e \in E(T)$.*

Suppose, for some ray R_i, the set $E(R_i) \setminus Y$ is finite. Let $Y' = E(R_i) \cap Y$. If t_i is adjacent to a leaf t_j of T, then M/Y' requires the deletion of all but a finite set of edges in $Q_{t_i t_j}$, contradicting 2.9.3. If e is not adjacent to a leaf of T, then M/Y' has multiple

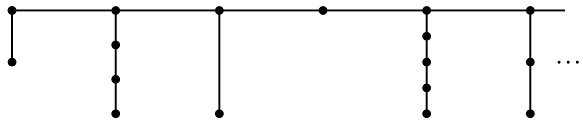

Fig. 4. Example of a comb graph.

ends, each containing at least one ray, and we take a cut set V' of M/Y' consisting of the star vertices and a vertex in each of the rays contained in one end. Clearly V' has fewer than c vertices, and each component of $(M/Y')\backslash V'$ is $(c-1)$-disconnected. Since N is ℓ-c-connected, by Lemma 2.4, it is not a minor of any component of $(M/Y') \setminus V'$. It is easy to see that it is also not a minor of M/Y'. We conclude with the following observation.

2.9.4. *For each ray R_i, the set $E(R_i) \setminus Y$ is infinite.*

By 2.9.4 and 2.9.2, for each ray R of the expansion M, there is a ray R' of the expansion N such that a subray of R' consists entirely of edges in R. Then N has m independent rays, hence it is not isomorphic to $K_{c,\infty}$. Also, N has no more than m independent rays, since the M has only m rays. Thus, by Lemma 2.6, these rays are indistinguishable from the rays of the expansion N. Take $R'_1, R'_2, \ldots, R'_m$ of the expansion N such that R'_i has its vertices contained entirely in R_i, for $i = 1, 2, \ldots, m$. Furthermore, 2.9.4, Lemmas 2.8 and 2.9 together imply that every component of G_Y is finite, though G_Y may contain infinitely many components, and no two stars of M are in a single component of G_Y. Thus, N has precisely $|S|$ vertices of infinite degree, each obtained by contracting a finite subgraph of M containing a star of M.

If we contract all of the edges in the m independent rays of N then the result is a graph with finitely many vertices. Let Z be its subgraph formed by edges from infinite parallel families. The simplification of Z must be isomorphic to T. For each edge t_it_j in T, by 2.9.3, Lemmas 2.8 and 2.9, t_it_j is an edge in Z. Graph N is therefore not an expansion of any tree other than T. $\square$

3. Unavoidable end behavior in locally finite infinite graphs

In this section we prove a result for augmenting paths, which will be essential for finding the unavoidable topological minors in locally finite ℓ-c-connected graphs. We begin with a stronger form of König's Infinity Lemma.

Lemma 3.1. *If G is a connected, locally finite infinite graph, then G contains an induced ray.*

Proof. Let G be a connected, locally finite infinite graph. Since G is locally finite, by Lemma 1.1, G has a ray $v_1v_2\cdots$. In addition, for each positive integer i, there exists the largest integer $n(i) > i$ such that v_i is adjacent to $v_{n(i)}$. It follows that $v_1v_{n(1)}v_{n(n(1))}\cdots$ is an induced ray of G. $\square$

A *comb* is a ray, the *spine* of the comb, combined with an infinite set of independent, finite paths, each containing exactly one vertex in the spine, as shown in Fig. 4. These finite paths are called *teeth*. Note that a ray is a comb, and all its vertices are teeth. The following theorem is proved in [1][8.2.2].

Theorem 3.2. *If $X_1, X_2, \ldots$ are pairwise-disjoint non-empty sets of vertices in a connected graph G, then G has either a comb containing a tooth that meets X_i for infinitely many of these sets or a subdivided star with leaves in infinitely many of these sets.*

An *end* of a graph G is an equivalence class of rays in G, where two rays are considered equivalent if, for every finite set $S \subset V(G)$, both have an infinite subray in the same component of $G \setminus S$.

The following theorem is a version of the main result of a paper by Halin [3]. We say that a finite vertex set U *separates* a vertex set V' in $V(G) \setminus U$ from an end ω in a graph G if the component of $G \setminus U$ containing the infinite component of each ray in ω does not meet V'. Halin's theorem deals with separating a single vertex from an end, and we give a slight variation of it here that is obtained by identifying a finite set of vertices V' in a locally finite graph and applying Halin's theorem to that single vertex.

Theorem 3.3. *For a locally finite graph G, the number of independent rays in one end of G that originate in a finite vertex set V' is equal to the cardinality of the smallest vertex set that separates V' from that end of G.*

We will also use the following lemma by Georgakopoulos [2]. We say that a set K of rays in an end ω of a graph G *devours* the end ω if every ray in ω meets a ray in K. An end devoured by a countable set of rays is *countable*.

Lemma 3.4. *For every graph G and every countable end ω of G, if G has a set K of k independent rays in ω, then it also has a set K' of k independent rays in ω that devours ω. Moreover, K' can be chosen so that its rays have the same starting vertices as the rays in K.*

We now state and prove the main result of this section, an essential theorem concerning locally finite ℓ-c-connected infinite graphs.

Theorem 3.5. *Suppose G is a locally finite, ℓ-c-connected graph, for some positive integer c. If G contains an end with $c-1$ independent rays, then G contains c independent rays in that end such that infinitely many vertices from each original ray are contained in the set of c rays.*

Proof. Observe that Lemma 1.1 implies the result when $c = 1$.

Let $R_1, R_2, \ldots, R_{c-1}$ be independent rays in an end ω of a locally finite, ℓ-c-connected graph G. Let X_i be the set of vertices consisting of the ith vertex of $R_1, R_2, \ldots, R_{c-2}$ and R_{c-1} plus the $(i+1)$th vertex of R_1 for each i in $\mathbb{N}$. Note that $|X_i| = c$. For all but finitely many i, the set X_i cannot be separated from the end ω by a separator comprising fewer than c vertices. Any such separator Y_i would have to contain one vertex y_j from each R_j, and then the initial part of R_j up to y_j, which has to contain the ith vertex of R_i, is in a finite component. If infinitely many such Y_i exist, then the graph is not ℓ-c-connected, since the sizes of those components are unbounded.

Take m large enough so that X_m cannot be separated from ω by a separator containing fewer than c vertices. By Theorem 3.3, ω contains c independent rays. The result follows from Lemma 3.4. $\square$

4. Unavoidable topological minors of c-connected infinite graphs

Let a graph G_1 be a subdivision of H_1, a member of $\mathcal{T}_c$, and let graph G_2 be a subdivision of a member of $\mathcal{T}_{c+1}$. We are interested in the case that G_2 contains the rays and star vertices of G_1, that is, the case that G_2 contains a copy of a subdivision of H_1 that is also contained in G_1. We say that G_2 is a *direct augmentation* of G_1, written $G_1^{\oplus}$, if G_2 contains a subgraph of G_1 that is itself a subdivision of H_1.

Note that any member G of $\mathcal{T}_{c+1}$ will contain a copy of one or more members of $\mathcal{T}_c$ as a series minor. Let S be this set of series minors. If, for example, G contains one ray and c star vertices, then S contains a member of $\mathcal{T}_c$ with c star vertices and a member of $\mathcal{T}_c$ with one ray and $c-1$ star vertices. Of the graphs in S, one or more will have the highest number of star vertices among the graphs in S. We now prove the following theorem, which implies Theorem 1.3(b).

Theorem 4.1. *For integer c at least two, let G be a ℓ-c-connected infinite graph. Take a subgraph D of G, where D is a subdivision of a graph in $\mathcal{T}_{c-1}$ with the maximal number of star vertices among all subgraphs of G. One of the following occurs:*

(i) *D contains a star vertex and G contains a graph $D^{\oplus}$; or*
(ii) *D is locally finite and G contains a graph Y that is a subdivision of a member of $\mathcal{T}_c$, such that Y contains infinitely many vertices from each ray of D.*

Proof. We will prove this theorem by induction on c. Let $c = 2$, and let G be a ℓ-c-connected infinite graph. Suppose G contains a vertex v adjacent to an infinite set S of vertices. Let D be the star with vertex set $S \cup \{v\}$. If $G - v$ contains a subdivision of a star with all of its leaves in S, then observe that G contains a subdivision of $K_{2,\infty}$, which itself contains an infinite subgraph of D and is a direct augmentation $D^{\oplus}$. Suppose not. We apply Theorem 3.2 to $N(v)$ in $G - v$ to obtain a comb C with infinitely many teeth that meet S. Observe that $D \cup C$ contains a subdivision of a fan, which is a direct augmentation of D.

If G has no vertex of infinite degree, then G is locally finite. We apply Lemma 1.1 to obtain D, a ray. We then apply Theorem 3.5 to D in G to obtain R_1 and R_2, independent rays in the end of G that contains infinitely many vertices in $V(D)$. We then apply Lemma 2.3 to R_1 and R_2 and the set of paths between them to obtain a subdivision of a ladder with poles contained in $R_1 \cup R_2$, and (ii) of the statement holds. We conclude that the theorem is true if $c = 2$.

Let $c = n$ for some integer n at least three, and let G be a ℓ-c-connected infinite graph. We assume that the statement holds for any number less than n. By the induction hypothesis, we may take D, a subdivision of a member of $\mathcal{T}_{c-1}$ with the maximal number of star vertices such that $D \subseteq G$. As an example, observe that any member of T_c that contains $k < c$ star vertices contains a subdivision of a member of T_{c-1} with k stars. The following two cases are exhaustive:

(1) D contains a vertex of infinite degree; or
(2) D is locally finite.

We need a bit of notation to addressing these cases. For any subdivision X of a member of $\mathcal{T}_i$, the *bag graphs* are the components of the graph after the deletion of the star vertices and the edges in each ray. If X contains a ray, then the bag graphs are ordered by the indices of that ray. If it contains no ray, then the bag graphs are ordered arbitrarily. The *bags* are the vertex sets of the bag graphs.

Suppose case (1) occurs and let v be a star vertex of D. We will show that we may augment a subgraph of $D - v$ that will form part of a direct augmentation of D. Let G_v be the subdivided star in G containing v such that each leaf has degree at least three in G and each interior vertex of G_v has degree two in G. Let D' be D after the deletion of the interior vertices of G_v and v. Graph D' is a subdivision of a member of $\mathcal{T}_{c-2}$. Furthermore, we claim the following.

4.1.1. *Graph D' has the maximal number of star vertices of all such subgraphs in the end of $G - v$ that contains D'.*

Suppose not. Then $G - v$ contains a subdivision H of a member of $\mathcal{T}_{c-2}$ with more star vertices than D' in the same end as D'. Then D' has at least one ray. Take a star vertex w in H that is not in D'. Since w is in the same end as D', G contains infinitely many independent paths between the neighbors of w and some ray R of D' such that the paths meet no other ray of D'. Then G contains a subdivision of a member of T_{c-1} that does not contain ray R but contains the star vertices in D' and v and w, which contradicts our choice of D. We conclude that 4.1.1 holds.

Since graph $G - v$ is ℓ-$(c-1)$-connected, we apply the induction assumption and conclude that $G - v$ contains a graph $D'^{\oplus}$ or $G - v$ contains a subdivision of a member of $\mathcal{T}_{c-1}$ that contains infinitely many vertices from each ray of D'. In either case, $G - v$ contains a graph Y such that Y is a subdivision of a member of $\mathcal{T}_{c-1}$ and Y contains vertices from infinitely many bags of D'. We may delete the edge sets of each bag graph that contains no vertex of Y, so without loss of generality, we assume that each bag meets Y.

We will now show that G contains a graph $Y^{\oplus}$ in $Y \cup G_v$.

As $\{V(G_v) \cap V(D')\}$ is infinite, G_v meets infinitely many bags of D'. Since we may delete some paths in G_v and the edge sets of some bag graphs in D', we assume without loss of generality that each leaf of G_v is contained in exactly one bag of D'. Let G_Y be the extension of the subdivided star G_v through the bag graphs such that $G_Y \cap Y$ is exactly the set of leaves of G_Y. Then G_Y contains infinitely many leaves in a ray R_i of Y, or G_Y contains infinitely many leaves in $Q_{t_it_j}$, the set of paths between star s_i or ray R_i and star s_j or ray R_j. Observe that $G_Y \cup Y$ contains a direct augmentation of Y that is also a direct augmentation of D, as desired.

It follows that G is locally finite, and we apply Theorem 3.5 to obtain c rays, $R_1, R_2, \ldots, R_c$, in G, which contain infinitely many vertices from each ray of D. We conclude this proof with the following lemma.

Lemma 4.2. *A series expansion of $(T, \emptyset)$, for some c-vertex tree T, is contained in G and has rays contained in $\{R_1 \cup R_2 \cup \cdots \cup R_c\}$.*

Proof. Between each pair of rays are infinitely many independent paths, since they are in the same end. Observe that some pair of rays, say R_1 and R_2, is joined by infinitely many independent paths that meet none of the other rays. Let H_1 be the subgraph of G containing R_1, R_2, and an infinite set Π_1 of independent paths that join them but meet none of the other rays. By Lemma 2.3, G has a ladder L_1 with poles R_1 and R_2 and rungs in Π_1. There is a ray, say R_3, such that G contains a set Π_2 of infinitely many independent paths between R_3 and L_1 that meet none of the remaining rays. Take a subset Π_2' of Π_2 such that L_1 contains infinitely many rungs that do not meet members of Π_2', but each of infinitely many paths in Π_2' meets a rung of L_1 or meets a pole, say R_1, of L_1. Each such path meeting a rung may be extended into a path that meets R_1. By Lemma 2.3, G has a ladder L_2 with poles R_1 and R_3 and rungs in Π_2'.

We continue in this fashion to attach ladder poles and rungs onto the graph, maintaining the pre-existing ladders, until all c rays have been attached. Observe that the resulting graph contains a subdivision of $(T, \emptyset)$ for some c-vertex tree, T. Furthermore, the rays of this graph are contained in $\{R_1 \cup R_2 \cup \cdots \cup R_c\}$, which contain infinitely many vertices from each ray of D, so H contains infinitely many vertices from each ray of D, and the lemma holds. It follows that Theorem 4.1 holds. $\square$

5. Unavoidable parallel minors of ℓ-c-connected infinite graphs

The following lemma is one application of "Ramsey's Theorem A" from [5].

Lemma 5.1. *If G is an infinite graph, then G has an induced subgraph isomorphic to K_∞ or $\overline{K_\infty}$.*

We conclude this paper with a proof of Theorem 1.4(b).

Proof of Theorem 1.4. Take a positive integer c. Let G be a ℓ-c-connected infinite graph that contains no minor isomorphic to K_∞. Graph G contains an infinite component, so we may ignore the finite components of G and assume that G is connected. We apply Theorem 1.2(b), a corollary of Theorem 1.3(b), to obtain a minor of G in $\mathcal{M}_c$. Let M be the minor of G in $\mathcal{M}_c$ containing the most star vertices and take edge sets X and Y such that $M = G \setminus X/Y$, where M spans G/Y.

If $M \cong K_{c,\infty}$, then we may add some edges to Y to obtain Y' such that $G \setminus X/Y' = M' \cong K'_{c,\infty}$. Since K_∞ is not a minor of G, we apply Lemma 5.1 to obtain an infinite independent set $A \subset V(G/Y')$. Let S be the set of star vertices in M'. Take $s \in S$. We contract the edges in G/Y' between s and each vertex in $V(M') \setminus \{S \cup A\}$ to obtain a parallel minor of G isomorphic to $K'_{c,\infty}$.

Suppose then that M is not isomorphic to $K_{c,\infty}$. Then M is an expansion of some tree T. Let S be the set of leaves of T. We add edges to Y to obtain Y' such that M/Y' is an expansion of (T, S). That is, $G \setminus X/Y'$ is isomorphic to the graph obtained from M by adding a complete graph on the star vertices, a vertex s_0 that is adjacent with each star and the first vertex of each ray, and a zigzag ladder between each pair of ladder poles in M. Now, let $M' = G \setminus X/Y'$. Take H, S, and T such that M' is an expansion of (H, S) and T is a leaf-maximal spanning tree of H with leaf set S. Consider the edges X in G/Y'.

For each vertex pair $\{t_i, t_j\}$ of $V(T)$, let $Q_{t_it_j}$ be the set of edges in G/Y' between R_i or s_i and R_j or s_j. We say that each edge in $Q_{t_it_j}$ is *between the vertex pair t_i and t_j*. Take integer n such that X contains edges between exactly n vertex pairs of $V(H)$. We prove the theorem by induction on n. If $n = 0$, then $X = \emptyset$ and an expansion of (H, S) is a parallel minor of G and the theorem holds. We assume that the theorem holds for $(n - 1)$.

Suppose that G/Y' contains edges between n vertex pairs of $V(H)$. Take one such vertex pair $\{t_i, t_j\}$.

If $Q_{t_it_j}$ is finite, then take a vertex r_l^k with highest index l for $k \in \{i, j\}$ that is incident with an edge in $Q_{t_it_j}$. Take a star vertex s of M'. For each ray R_a, we contract the path $sr_1^ar_2^a \ldots r_l^a$ to vertex s to eliminate the edges in $Q_{t_it_j}$ and obtain a minor $G \setminus X'/Y''$ of $G \setminus X/Y'$ that contains a copy of M'. Then X' contains edges between fewer than n vertex pairs of $V(H)$. We apply the inductive hypothesis and conclude that the theorem holds.

Suppose then that $Q_{t_it_j}$ is infinite. The following three cases are exhaustive:

(1) $t_i = R_i = t_j$;
(2) $t_i = R_i$ and $t_j = s_j$; or
(3) $t_i = R_i \neq t_j = R_j$.

For the rest of the proof, it will be convenient to let $E(r_lr_{l+1})$ denote the edge set $\{r_l^kr_{l+1}^k : R_k \text{ is a ray of } M'\}$.

Suppose case (1) occurs. Let R' be the graph that $Q_{t_it_j}$ induces on $V(R_i)$. If R' contains a vertex r of infinite degree, then we contract the edge sets $E(r_lr_{l+1})$ if and only if $r_l^i \notin N(r)$, where $N(r)$ is the neighborhood of vertex r. Observe that r is a star of the resulting graph, thus G contains a minor in $\mathcal{M}_c$ with more star vertices than M, a contradiction. We make the following observation, where S is the set of stars of M'.

5.1.1. *The graph that the edge set $Q_{t_it_j}$ induces in $M' \setminus S$ is locally finite.*

If R' is locally finite, then let $r_1^i = r_{n_1}$. Let r_{n_2} be the vertex with highest index among the neighbors of r_{n_1} in R'. Let r_{n_i} be the vertex with highest index that is a neighbor of a vertex in the path $r_{n_{i-2}}r_{n_{i-2}+1}\dots r_{n_{i-1}}$. We contract the edge set $E(r_lr_{l+1})$ if and only if $l \notin \{n_1, n_2, \dots\}$. By these contractions in R', we contract each edge of $Q_{t_it_j}$ to a single vertex. In this way, we obtain a parallel minor of G that contains a copy of M' and the remaining edges of X are between at most $(n-1)$ vertex pairs of $V(H)$. We apply the inductive hypothesis and conclude that the theorem holds. We therefore assume that case (1) does not occur.

Suppose case (2) occurs: $t_i = R_i$ and $t_j = s_j$. We contract the edge set $E(r_lr_{l+1})$ if and only if $l \notin N(s_j)$ to obtain an expansion of $(H \cup t_it_j, S)$. Tree T is a leaf-maximal spanning tree, and we obtain a parallel minor of G that contains a copy of M' and the remaining edges of X are between at most $(n-1)$ vertex pairs of $V(H)$ not in $E(H \cup \{t_it_j\})$. We apply the inductive hypothesis and conclude that the theorem holds. We also make the following observation.

5.1.2. *If a star s is adjacent with infinitely many vertices in a ray R_i in Z, then we may assume s to be adjacent with every vertex in R_i.*

We therefore assume that case (2) does not occur.

Suppose case (3) occurs: $t_i = R_i \neq t_j = R_j$. We apply 5.1.1 and conclude that $Q_{t_it_j}$ contains no infinite set of edges adjacent with a single vertex, thus $Q_{t_it_j}$ contains an infinite set Π of pairwise non-adjacent edges.

The following argument is technical and amounts to obtaining a zigzag ladder on R_i and R_j. We break up the edge set $E(r_lr_{l+1})$ into two sets. Edge t_it_j is a cut set of tree T and divides the graph into a component containing t_i and a component containing t_j. Let $E_i(r_lr_{l+1})$ be the set of edges corresponding to the edges in $E(r_lr_{l+1})$ that are in the rays labelling vertices in the component of $T \setminus t_it_j$ containing t_i. Let $E_j(r_lr_{l+1})$ be the set of edges $E(r_lr_{l+1}) \setminus E_i(r_lr_{l+1})$. We apply Lemma 2.3 to obtain L, a subdivided ladder with poles in R_i and R_j and with rung set ρ in Π. This allows us to assume that, for every integer $k > 0$, we may find a rung in ρ with ends in the infinite components of $R_i - r_k^i$ and $R_j - r_k^j$. Let $i_1 = 1$. Let j_1 be the lowest index such that $r_1^jr_2^j\dots r_{j_1}^j$ has a neighbor in $R_i - r_1^i$ and $j_1 \geq m$ for each vertex r_m^j adjacent with $r_{i_1}^i$. For $n = 2, 3, \dots$, let i_n be the lowest index such that $i_n > m$ for each vertex r_m^i adjacent with a vertex in $r_1^jr_2^j\dots r_{j_{n-1}}^j$ and $r_{i_{n-1}+1}^ir_{i_{n-1}+2}^i\dots r_{i_n}^i$ has a neighbor in the infinite component of $R_j - r_{n-1}^j$; and let j_n be the lowest index such that $j_n \geq m$ for each vertex r_m^j adjacent with a vertex in $r_1^ir_2^i\dots r_{i_n}^i$ and $r_{j_{n-1}+1}^jr_{j_{n-1}+2}^j\dots r_{j_n}^j$ has a neighbor in the infinite component of $R_i - r_n^i$. Contract edge set $E_i(r_lr_{l+1})$ if and only if $l \notin \{i_1, i_2, \dots\}$ and contract edge set $E_j(r_lr_{l+1})$ if and only if $l \notin \{j_1, j_2, \dots\}$ to obtain a zigzag ladder on R_i and R_j. Let Z be the resulting graph. The graph that Z induces on rays R_i and R_j is a zigzag ladder.

If $t_it_j \in E(T)$, then Z is an expansion of (H, S), and the theorem holds.

If $t_it_j \notin E(T)$, then $T \cup R_iR_j$ contains a cycle $C = R_{k_1}R_{k_2}\dots R_{k_l}$ of interior vertices, where $k_1 = i$ and $k_2 = j$. Observe that T is not leaf-maximal in $H \cup t_it_j$. We will show that G contains a member of $\mathcal{M}_c$ with more star vertices than M and obtain a contradiction. We begin by identifying a set of l rays in Z each of which contains infinitely many vertices of each ray in this cycle. Since there are two different ways of expressing a zigzag ladder between two rays, we will have to be careful with this construction. Let $\phi(a)$ be equal to one if $r_1^{k_a}r_2^{k_{a+1}} \in E(Z)$, where we say that $l+1 = 1$, otherwise $\phi(a) = 0$. Let $\Sigma(a) = 1 + \sum_{m=1}^{a}\phi(m)$. Let ray R_1' be $r_1^{k_1}r_{\Sigma(1)}^{k_2}r_{\Sigma(2)}^{k_3}r_{\Sigma(3)}^{k_4}\cdots$. For $m = 2, 3, \dots, l$, let

$$R_m' = r_1^{k_m}r_{\Sigma(m)}^{k_{m+1}}r_{\Sigma(m+1)}^{k_{m+2}}r_{\Sigma(m+2)}^{k_{m+3}}\cdots.$$

Observe that these l rays are independent and each contains infinitely many vertices of each of the l original rays of Z. The graph that Z induces on each pair of rays R_m' and R_{m+1}', where $l+1 = 1$, is a zigzag ladder. We also conclude the following.

5.1.3. *Every ray and star labelling a vertex of H with infinitely many neighbors in R_1' contains infinitely many neighbors in R_m' for $m = 2, 3, \dots, l$.*

We will now show that G contains a minor in $\mathcal{M}_c$ with more star vertices than M, a contradiction that will conclude our proof.

Let S_Z be the star set of Z. We will show that R_1' is not a cut set of $Z \setminus S_Z$ and that no star has infinitely many neighbors only in R_1' and conclude that we may contract R_1' without losing ℓ-c-connectivity. Let R be a ray containing infinitely many vertices adjacent with R_1'. Apply 5.1.3 and conclude that R has infinitely many neighbors in R_2'. We apply Lemma 2.3 and conclude that the graph that Z induces on $R \cup R_1'$ contains a subdivision of a ladder. Let s be a star with infinitely many neighbors in R_1'. We apply 5.1.3 and conclude that s is adjacent to an infinite subset of vertices in R_2', and we may apply 5.1.2 to this pair and assume that s is adjacent to each vertex in R_1'. We contract ray R_1' in Z to obtain an ℓ-c-connected graph that contains a member of $\mathcal{M}_c$ with more star vertices than M, a contradiction. We may assume that case (3) does not occur. □

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